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Penalty combinations of the Ritz–Galerkin and finite difference methods for singularity problems[1]

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Abstract

Penalty combination of the Ritz–Galerkin and finite difference methods is presented for solving elliptic boundary value problems with singularities. The superconvergence rate, $O(h^{2-\delta})$, of solution derivatives by the combination can be achieved while using quasiuniform rectangular difference grids, where h is the maximal mesh length of difference grids used in the finite difference method, and $\delta(>0)$ is an arbitrarily small number. It is due to its simplicity that the penalty combination of the Ritz–Galerkin and finite difference methods is highly recommended for solving the complicated problems with multiple singularities and multiple interfaces.

Keywords: Superconvergence; Combined method; Coupling strategy; Finite elliptic equation; Singularity problem; Superconvergence; Combined method; Coupling strategy; Finite difference method; Ritz–Galerkin method; Penalty method

AMS classification: 65N10, 65N30

1. Introduction

Using different numerical methods is important for solving the complicated problems of elliptic equations, in particular those with multiple singularities and multiple interfaces. In this paper we will study superconvergence of solution derivatives by the penalty combination of the Ritz–Galerkin method (i.e. using analytical and singular functions), and the finite difference method (simply written the RG-FDMs).

It is worth pointing out that superconvergence of solutions obtained from *single* methods is given in many reports, such as in [10–14, 1–4,16], in particular in the monographs [9, 15, 17]. We shall

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focus on analysis on the coupling techniques and their incorporation with the finite difference method. Therefore, other treatments on superconvergence rates of finite element method can be matched into penalty combinations as well.

Superconvergence rate $O(h^{2-\delta})$ of solution derivatives is proven in [5] by the nonconforming combination for Motz's problem using uniform squares, where h is the boundary length, and $\delta(>0)$ is an arbitrarily small number. The nonconforming combination incurs a trouble in dealing with the constraints, by which two different admissible functions are matched along the common boundary Γ_0. Hence, the penalty techniques along Γ_0 are adopted to bypass the trouble, thus to lead to the penalty combination. In [6], the penalty combination of the Ritz–Galerkin and finite element methods is explored, to prove that quasiuniform triangulation without especial posterior treatments grants the optimal convergence rate $O(h)$ of solution derivatives if the penalty parameter $\sigma=2$ used. In this paper, the superconvergence rate $O(h^{2-\delta})$ can be achieved by the penalty combination of RG-FDMs for general elliptic boundary value problems using quasiuniform rectangles if $\sigma \geqslant 4$, where h is the maximal mesh spacing of difference grids. Besides, only a lower order of $O(h^{3/2})$ for solution derivatives can be obtained if quasiuniform triangular difference elements in [5] are also chosen near the boundary. However, if applying the treatments of triangular elements in linear finite element methods, e.g., such as in [3,4,9–16], the superconvergence rates, $O(h^{2-\delta})$, can also be regained.

2. The penalty combinations of the RG-FDMs

Consider the Poisson equation with the Dirichlet boundary condition

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(x, y), \quad (x, y) \in S, \tag{2.1}$$

$$u = 0, \quad (x, y) \in \Gamma, \tag{2.2}$$

where S is a polygon domain, Γ the exterior boundary ∂S of S, and f the smooth enough. Let the solution domain S be divided by a piecewise straight line Γ_0 into two subdomains S_1 and S_2. The Ritz–Galerkin method is used in S_2 where there may exist a singular point, and the finite difference method is used in S_1. The subdomain S_1 is again split by difference grids into small rectangles $\square_{ij}$ and triangles $\triangle_{ij}$. Denote $u_{i,j} = u(x_i, y_j)$, where (i, j) or (x_i, y_j) denotes the location of difference nodes. Assume that the difference grids in S_1 are quasiuniform, i.e., there exists a bounded constant C independent of h_i and k_j such that $h/\min_{i,j}(h_i, k_j) \leqslant C$, where $h_i = x_{i+1} - x_i$, $k_j = y_{j+1} - y_j$, and the maximal mesh spacing $h = \max_{i,j}(h_i, k_j)$. The conventional finite difference method can be regarded as a special kind of finite element methods using piecewise bilinear and linear interpolatory functions $v_1(x, y)$ on $\square_{ij}$ and $\triangle_{ij}$, respectively,

$$\begin{aligned} v_1(x, y) = \frac{1}{h_i k_j} \{&(x_{i+1} - x)(y_{j+1} - y)v_{ij} + (x - x_i)(y_{j+1} - y)v_{i+1,j} \\ &+ (x_{i+1} - x)(y - y_j)v_{i,j+1} + (x - x_i)(y - y_j)v_{i+1,j+1})] \quad \text{for } (x, y) \in \square_{ij}, \end{aligned} \tag{2.3}$$

and

$$v_1(x,y) = v_{ij} + \frac{(x-x_i)}{h_i}(v_{i+1,j} - v_{i,j}) + \frac{(y-y_j)}{k_j}(v_{i,j+1} - v_{ij}), \quad \text{for } (x,y) \in \triangle_{ij}. \tag{2.4}$$

The boundary difference nodes (i,j) are placed on ∂S_1 and the triangles $\triangle_{ij}$ are always located near the boundary ∂S_1 of S_1. Hence, the total number of $\triangle_{ij}$ is much less than that of $\square_{ij}$.

In S_2, we assume that the solution u can be spanned by $u = \psi_0 + \sum_{i=1}^{\infty} a_i \psi_i$, where a_i are the expansion coefficients, and ψ_i $(i = 1, 2, \ldots, \infty)$ are complete and linearly independent basis functions, which may be chosen as analytical and singular functions. Then the admissible functions of combinations of the RG-FDMs are written as

$$v = \begin{cases} v^- = v_1, & \text{in } S_1, \\ v^+ = f_L(\tilde{a}_i) & \text{in } S_2, \end{cases} \tag{2.5}$$

where $\tilde{a}_i$ are unknown coefficients to be sought, and

$$f_L(a_i) = \psi_0 + \sum_{i=1}^{L} a_i \psi_i. \tag{2.6}$$

We define another space $H = \{v \mid v \in L^2(S),\ v \in H^1(S_1), \text{ and } v \in H^1(S_2)\}$, where $H^1(S_1)$ is the Sobolev space. Let $\bar{V}_h (\subseteq H)$ denote a finite-dimensional collection of the function v in (2.5) satisfying (2.2). The penalty combinations of the RG-FDMs involving integral approximation on Γ_0 can be expressed by

$$\hat{a}_h(u_h, v) = \hat{f}_h(v), \quad \forall v \in \bar{V}_h, \tag{2.7}$$

where

$$\hat{a}_h(u,v) = \widehat{\iint_{S_1}} \nabla u \nabla v \, \mathrm{d}s + \iint_{S_2} \nabla u \nabla w \, \mathrm{d}s + \frac{P_c}{h^\sigma} \widehat{\int_{\Gamma_0}} (u^+ - u^-)(v^+ - v^-) \, \mathrm{d}\ell, \tag{2.8}$$

$P_c (> 0)$ is the penalty constant, σ is the penalty power, and

$$\widehat{\iint_{S_1}} \nabla u \nabla v \, \mathrm{d}s = \sum_{ij} \left[\widehat{\iint_{\square_{ij}}} \nabla u \nabla v \, \mathrm{d}s + \widehat{\iint_{\triangle_{ij}}} \nabla u \nabla v \, \mathrm{d}s \right], \tag{2.9}$$

$$\hat{f}_h(v) = \widehat{\iint_{S_1}} f\, v \, \mathrm{d}s + \iint_{S_2} f\, v \, \mathrm{d}s, \tag{2.10}$$

$$\hat{f}_1(v) = \widehat{\iint_{S_1}} f\, v \, \mathrm{d}s = \sum_{ij} \left[\widehat{\iint_{\square_{ij}}} f\, v \, \mathrm{d}s + \widehat{\iint_{\triangle_{ij}}} f\, v \, \mathrm{d}s \right]. \tag{2.11}$$

The approximate integrals in (2.9) and (2.11) are evaluated by the following specific rules:

$$\widehat{\iint_{\square_{ij}}} \nabla u \nabla v \, \mathrm{d}s = \widehat{\iint_{\square_{ij}}} u_x v_x \, \mathrm{d}s + \widehat{\iint_{\square_{ij}}} u_y v_y \, \mathrm{d}s, \tag{2.12}$$

$$\widehat{\iint_{\square_{ij}}} u_x v_x \, \mathrm{d}s = \tfrac{1}{2} h_i k_j \, [u_x(i+\tfrac{1}{2}, j) v_x(i+\tfrac{1}{2}, j) + u_x(i+\tfrac{1}{2}, j+1) v_x(i+\tfrac{1}{2}, j+1)], \tag{2.13}$$

$$\widehat{\iint_{\square_{ij}}} u_y v_y \, ds = \tfrac{1}{2} h_i k_j [u_y(i, j+\tfrac{1}{2}) v_y(i, j+\tfrac{1}{2}) + u_y(i+1, j+\tfrac{1}{2}) v_y(i+1, j+\tfrac{1}{2})], \tag{2.14}$$

$$\widehat{\iint_{\square_{ij}}} f v \, ds = \tfrac{1}{4} h_i k_j \, [f_{ij} v_{ij} + f_{i+1,j} v_{i+1,j} + f_{i,j+1} v_{i,j+1} + f_{i+1,j+1} v_{i+1,j+1}], \tag{2.15}$$

where $u_x(i+\frac{1}{2}, j) = u_x(x_{i+1/2}, y_j)$, $x_{i+1/2} = \frac{1}{2}(x_i + x_{i+1})$, and the rectangle $\square_{ij} = \{(x, y), x_i \leqslant x \leqslant x_{i+1}, y_j \leqslant y \leqslant y_{j+1}\}$. For the right triangle $\triangle_{ij}$

$$\widehat{\iint_{\triangle_{ij}}} \nabla u \nabla v \, ds = \widehat{\iint_{\triangle_{ij}}} (u_x v_x + u_y v_y) \, ds$$

$$= \tfrac{1}{2} h_i k_j [u_x(i+\tfrac{1}{2}, j) v_x(i+\tfrac{1}{2}, j) + u_y(i, j+\tfrac{1}{2}) v_y(i, j+\tfrac{1}{2})], \tag{2.16}$$

$$\widehat{\iint_{\triangle_{ij}}} f v \, ds = \tfrac{1}{8} h_i k_j (2 f_{ij} v_{ij} + f_{i+1,j} v_{i+1,j} + f_{i,j+1} v_{i,j+1}). \tag{2.17}$$

For the integral along Γ_0 in (2.8), letting $\Gamma_0 = \bigcup_{k=1}^{N_1} \Gamma_0^{(k)}$, $\Gamma_0^{(k)} = Z_{k-1} Z_k$, we choose the integration rules.

$$\int_{\Gamma_0} \xi \eta \, d\ell \approx \widehat{\int_{\Gamma_0}} \xi \eta \, d\ell = \int_{\Gamma_0} \hat{\xi} \hat{\eta} \, d\ell$$

$$= \sum_{k=1}^{N_1} \frac{\overline{Z_{k-1} Z_k}}{6} [2\xi(Z_{k-1}) \eta(Z_{k-1}) + \xi(Z_{k-1}) \eta(Z_k)$$

$$+ \xi(Z_k) \eta(Z_{k-1}) + 2\, \xi(Z_k) \eta(Z_k)], \tag{2.18}$$

where $\overline{Z_{k-1} Z_k}$ denotes the length of $Z_{k-1} Z_k$, and $\hat{\xi}$ and $\hat{\eta}$ are the piecewise linear interpolatory functions along Γ_0.

3. Superconvergence analysis

Norm definitions are needed for evaluating error bounds of solutions obtained by the combinations. We thus define

$$\|v\|_h = \left(\|v\|_{1,S_1}^2 + \|v\|_{1,S_2}^2 + \frac{P_c}{h^\sigma} \|v^+ - v^-\|_{0,\Gamma_0}^2 \right)^{1/2}, \tag{3.1}$$

$$\|v\|_1 = (\|v\|_{1,S_1}^2 + \|v\|_{1,S_2}^2)^{1/2}, \tag{3.2}$$

where the Sobolev norms are given by

$$\|u\|_{m,\Omega} = \left(\sum_{|n| \leqslant m} \iint_\Omega |D^n u|^2 \, d\Omega \right)^{1/2}, \qquad |v|_{m,\Omega} = \left(\sum_{|n| = m} \iint_\Omega |D^n u|^2 \, d\Omega \right)^{1/2}. \tag{3.3}$$

Optimal convergence rates of numerical solutions

$$\|\varepsilon\|_h = \|u - u_h\|_h = \mathrm{O}(h) \tag{3.4}$$

can be obtained similarly by means of [6].

In this paper, we pursue superconvergence based on the new norms:

$$\overline{\|v\|}_h = \left(\overline{\|v\|}^2_{1,S_1} + \|v\|^2_{1,S_2} + \frac{P_c}{h^\sigma}\,\overline{\|v^+ - v^-\|}^2_{0,\Gamma_0}\right)^{1/2}, \tag{3.5}$$

$$\overline{\|v\|}_1 = (\overline{\|v\|}^2_{1,S_1} + \|v\|^2_{1,S_2})^{1/2}, \tag{3.6}$$

where the norms with discrete summation are

$$\overline{\|v\|}^2_{1,S_1} = \overline{|v|}^2_{1,S_1} + \overline{\|v\|}^2_{0,S_1}, \tag{3.7}$$

$$\overline{|v|}^2_{1,S_1} = \sum_{ij}\left[\widehat{\iint_{\square_{ij}}} (\nabla v)^2\,\mathrm{d}s + \widehat{\iint_{\triangle_{ij}}} (\nabla v)^2\,\mathrm{d}s\right], \tag{3.8}$$

$$\overline{\|v\|}^2_{0,S_1} = \sum_{ij}\left[\widehat{\iint_{\square_{ij}}} v^2\,\mathrm{d}s + \widehat{\iint_{\triangle_{ij}}} v^2\,\mathrm{d}s\right]. \tag{3.9}$$

The discrete formulas, $\widehat{\iint_{\square_{ij}}}(\nabla v)^2\,\mathrm{d}s$, $\widehat{\iint_{\triangle_{ij}}}(\nabla v)^2\,\mathrm{d}s$, $\widehat{\iint_{\square_{ij}}} v^2\,\mathrm{d}s$ and $\widehat{\iint_{\triangle_{ij}}} v^2\,\mathrm{d}s$, are given by (2.12)–(2.17). Also, the norm on Γ_0 is defined by

$$\overline{\|v^+ - v^-\|}^2_{0,\Gamma_0} = \widehat{\int_{\Gamma_0}} (v^+ - v^-)^2\,\mathrm{d}\ell, \tag{3.10}$$

where the integration $\widehat{\int_{\Gamma_0}} v^2\,\mathrm{d}\ell$ is given in (2.18). Notice that the definition of $\overline{\|v\|}_h$ perfectly agrees with the integration rules used in the penalty combinations, and this norm will play an important role in obtaining superconvergence of the solutions.

The superconvergence rates of

$$\overline{\|\varepsilon\|}_h = \mathrm{O}(h^{2-\delta}) \tag{3.11}$$

and

$$\overline{\|\varepsilon\|}_h = \mathrm{O}(h^{3/2}) \tag{3.12}$$

can be achieved by penalty combinations for the quasiuniform partitions:

$$S_1 = \cup_{ij} \square_{ij} \tag{3.13}$$

and

$$S_1 = (\cup_{ij} \square_{ij}) \cup (\cup_{ij} \triangle_{ij}), \tag{3.14}$$

respectively. Let us prove these conclusions. First, the following error bounds of solution u_h can be obtained from (2.7), by following [5, 6] (a detailed proof is given in [8]):

$$\overline{\|u-u_h\|}_h \leqslant C\left\{\inf_{v\in\bar V_h}\overline{\|u-u_h\|}_h+\sup_{w\in\bar V_h}\frac{|(\iint_{S_1}-\widehat{\iint}_{S_1})\nabla u\,\nabla w\,\mathrm{d}s|}{\overline{\|w\|}_h}\right.$$
$$\left.+\sup_{w\in\bar V_h}\frac{|(\iint_{S_1}-\widehat{\iint}_{S_1})f\,w\,\mathrm{d}s|}{\overline{\|w\|}_h}+\sup_{w\in\bar V_h}\frac{|(\int_{\Gamma_0}\frac{\partial u}{\partial n}(w^+-w^-)\,\mathrm{d}\ell|}{\overline{\|w\|}_h}\right\}. \tag{3.15}$$

Let us now prove the following lemmas, to estimate the bounds of all terms in (3.15).

Lemma 3.1. *Let*

$$|v^+|_{\ell,\Gamma_0}\leqslant CL^{\ell\mu}\,\|v^+\|_{0,\Gamma_0},\quad \ell=1,2,\ \forall v\in\bar V_h, \tag{3.16}$$

hold, where $\mu(>0)$ is a bounded power independent of L,h and v. Then

$$\|v^+-v^-\|_{0,\Gamma_0}\leqslant\overline{\|v^+-v^-\|}_{0,\Gamma_0}+C(hL^\mu)^2\,\|v\|_{1,S_2},\quad \forall\, v\in\bar V_h. \tag{3.17}$$

Proof. Since $\overline{\|w\|}_{0,\Gamma_0}=\|\hat w\|_{0,\Gamma_0}$, we have from the triangular inequality

$$\|w\|_{0,\Gamma_0}-\overline{\|w\|}_{0,\Gamma_0}=\|w\|_{0,\Gamma_0}-\|\hat w\|_{0,\Gamma_0}\leqslant\|w-\hat w\|_{0,\Gamma_0}. \tag{3.18}$$

Letting $w=v^+-v^-$ we obtain from (3.18)

$$\begin{aligned}\|v^+-v^-\|_{0,\Gamma_0}-\overline{\|v^+-v^-\|}_{0,\Gamma_0}&\leqslant\|v^+-\hat v^+\|_{0,\Gamma_0}\leqslant Ch^2\,|v^+|_{2,\Gamma_0}\\&\leqslant Ch^2L^{2\mu}\,|v^+|_{0,\Gamma_0}\leqslant Ch^2L^{2\mu}\,|v|_{1,S_2}.\end{aligned} \tag{3.19}$$

This completes the proof of Lemma 3.1. □

From Lemma 3.1 we obtain the bounds of one term in (3.15):

$$\begin{aligned}\left|\int_{\Gamma_0}\frac{\partial u}{\partial n}(w^+-w^-)\,\mathrm{d}\ell\right|&\leqslant\left\|\frac{\partial u}{\partial n}\right\|_{0,\Gamma_0}\|w^+-w^-\|_{0,\Gamma_0}\\&\leqslant\left\|\frac{\partial u}{\partial n}\right\|_{0,\Gamma_0}\left[\overline{\|w^+-w^-\|}_{0,\Gamma_0}+C(hL^\mu)^2\,\|w\|_{1,S_2}\right]\\&\leqslant C\left\|\frac{\partial u}{\partial n}\right\|_{0,\Gamma_0}[h^{\sigma/2}+(hL^\mu)^2]\overline{\|w\|}_h.\end{aligned} \tag{3.20}$$

Suppose that the piecewise straight lines Γ_0^* $(\subset S_1)$ consist of the difference coordinate lines, by which S_1 is divided into S_1^* and S_0^* (see Fig. 1), i.e., $S_1=S_1^*\cup S_0^*$. The middle region S_0^* is between S_1^* and S_2 so that there always exists a distance between S_1^* and S_2, i.e., $\mathrm{Dist}(S_1^*,S_2)>0$. Also let $\phi(x,y)\in[0,1]$ be an analytic function on S such that

$$\phi(x,y)=\begin{cases}1 & \text{in } S_1^*,\\ 0 & \text{in } S_2.\end{cases} \tag{3.21}$$

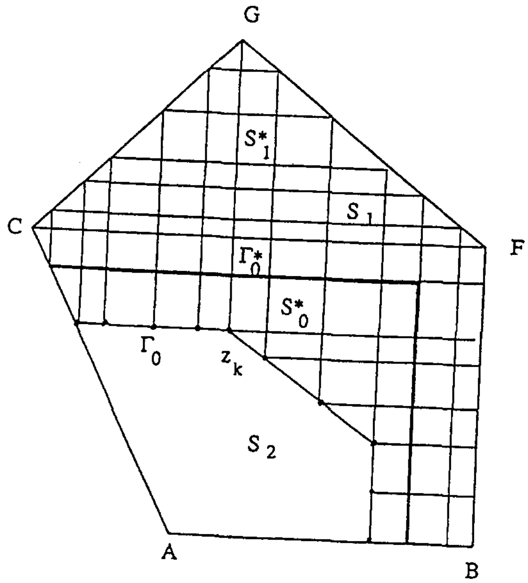

Fig. 1. Quasiuniform difference grids in S_1.

Define an auxiliary function

$$\bar{u} = \begin{cases} u & \text{in } S_1^*, \\ f_L(a_\ell) + \phi(x,y) \sum_{i=L+1}^{\infty} a_i \psi_i & \text{in } S_0^*, \\ f_L(a_\ell) & \text{in } S_2, \end{cases} \tag{3.22}$$

where $f_L(a_\ell) = \psi_0 + \sum_{\ell=1}^{L} a_\ell \psi_\ell$, and a_ℓ are the true coefficients. Construct a particular admissible function $\bar{w}_h \in \bar{V}_h$ such that

$$\bar{w}_h = \begin{cases} \bar{u}_1 & \text{in } S_1, \\ f_L(a_\ell) & \text{in } S_2, \end{cases} \tag{3.23}$$

where $\bar{u}_1$ is the piecewise bilinear and linear interpolatory function of $\bar{u}$ on the difference partition of S_1. We have the following lemma.

Lemma 3.2. *Let*

$$u \in C^3(S_1) \tag{3.24}$$

hold, where $C^k(S_1)$ denotes the space of functions having k-order continuous derivatives. Then

$$\inf_{v \in \bar{V}_h} \overline{\|u - v\|_1} \leqslant Ch^2 + \|R_L\|_{1,S_2 \cup S_0^*}, \tag{3.25}$$

where the remainder $R_L = \sum_{\ell=L+1}^{\infty} a_\ell \psi_\ell$.

Proof. We have from the triangular inequality

$$\inf_{v\in\bar{V}_h} \overline{\|u-v\|}_1 \leqslant \overline{\|u-\bar{w}_h\|}_1 \leqslant \overline{\|u-\bar{u}\|}_1 + \overline{\|\bar{u}-\bar{w}_h\|}_1, \tag{3.26}$$

where $\bar{w}_h(\in \bar{V}_h)$ is given in (3.23). By noting the definition of $\bar{u}$ in (3.22), we obtain

$$\overline{\|u-\bar{u}\|}_1 = \|u-\bar{u}\|_{1,S_2\cup S_0^*} \leqslant \|R_L\|_{1,S_2\cup S_0^*}. \tag{3.27}$$

The function $\bar{u}\in C^3(S_1)$ due to (3.22) and (3.24). Then letting $\delta=\bar{u}-\bar{w}_h$, we can see from the definition (3.7) and $\overline{\|\delta\|}_{0,S_1}=0$ that

$$\begin{aligned}\overline{\|\delta\|}_1^2 = \overline{\|\delta\|}_{1,S_1}^2 &= \overline{|\delta|}_{1,S_1}^2\\ &= \sum_{\forall\Box_{ij}} \tfrac{1}{2}h_i k_j\,[\delta_x^2(i+\tfrac{1}{2},j)+\delta_x^2(i+\tfrac{1}{2},j+1)+\delta_y^2(i,j+\tfrac{1}{2})+\delta_y^2(i+1,j+\tfrac{1}{2})]\\ &\quad + \sum_{\forall\triangle_{ij}} \tfrac{1}{2}h_i k_j\,[\delta_x^2(i+\tfrac{1}{2},j)+\delta_y^2(i,j+\tfrac{1}{2})] \leqslant Ch^4\sum_{ij} h_i kj\, M_3^2(\bar{u}) \leqslant Ch^4,\end{aligned} \tag{3.28}$$

where $M_n(u)=\max_{i+j=k\leqslant n,(x,y)\in S_1} |\partial^k u/\partial x^i\partial y^j|$. The desired result (3.25) is obtained from (3.26)–(3.28); this completes the proof of Lemma 3.2. □

Below we shall prove another important lemma.

Lemma 3.3. *Let* (3.16), (3.24) *and*

$$f\in C^2(S_1) \tag{3.29}$$

hold. Suppose (3.13), *i.e.,* S_1 *consists of only quasiuniform rectangles. Then there exist the bounds*

$$\left|\left(\iint_{S_1} - \widehat{\iint_{S_1}}\right) \nabla u\,\nabla w\,\mathrm{d}s\right| \leqslant C\{h^2L^\mu + h^{1+\sigma/2}\}M_3(u)\,\overline{\|w\|}_1,\ \ \forall w\in\bar{V}_h, \tag{3.30}$$

$$\left|\left(\iint_{S_1} - \widehat{\iint_{S_1}}\right) f\,w\,\mathrm{d}s\right| \leqslant Ch^2M_2(f)\,\overline{\|w\|}_1,\ \ \forall w\in\bar{V}_h. \tag{3.31}$$

Proof. Since

$$\left|\left(\iint_{S_1} - \widehat{\iint_{S_1}}\right) \nabla u\,\nabla w\,\mathrm{d}s\right| \leqslant \left|\left(\iint_{S_1} - \widehat{\iint_{S_1}}\right) u_x\,w_x\,\mathrm{d}s\right| + \left|\left(\iint_{S_1} - \widehat{\iint_{S_1}}\right) u_y\,w_y\,\mathrm{d}s\right|, \tag{3.32}$$

we only prove bounds of one term on the right-hand side of (3.32), for example,

$$\left|\left(\iint_{S_1} - \widehat{\iint_{S_1}}\right) u_x\,w_x\,\mathrm{d}s\right| \leqslant C\{h^2L^\mu + h^{1+\sigma/2}\}M_3(u)\overline{\|w\|}_1,\ \ \forall w\in\bar{V}_h, \tag{3.33}$$

because the proof for bounds of the other term is the same. By using Taylor's formula we obtain

$$\iint_{\square_{ij}} g \, \mathrm{d}s = \tfrac{1}{2} h_i k_j \left[g(i+\tfrac{1}{2}, j) + g(i+\tfrac{1}{2}, j+1) \right] + R_{i,j}^{(1)}, \tag{3.34}$$

where the truncation errors

$$R_{i,j}^{(1)} = h_i k_j \left\{ \frac{1}{24} \left(h_i^2 \frac{\partial^2 \tilde{g}_{ij}^{(1)}}{\partial x^2} - 2k_j^2 \frac{\partial^2 \tilde{g}_{ij}^{(2)}}{\partial y^2} \right) + \frac{1}{32} h_i k_j \left[\frac{\partial^2 \tilde{g}_{ij}^{(3)}}{\partial x \partial y} - \frac{\partial^2 \tilde{g}_{ij}^{(4)}}{\partial x \partial y} \right] \right\}, \tag{3.35}$$

$$\tilde{g}_{ij}^{(k)} = g(\xi_{ij}^{(k)}), \quad \xi_{ij}^{(k)} \in \square_{ij}, \quad k = 1,2,3,4. \tag{3.36}$$

Since $w (\in \bar{V}_h)$ is a bilinear function on $\square_{ij}$, then we have from (2.3)

$$w_{xx} = w_{yy} = 0 \quad \text{in } \square_{ij}, \tag{3.37}$$

$$w_{xy} = \frac{1}{h_i k_j} [w_{ij} - w_{i+1,j} - w_{i,j+1} + w_{i+1,j+1}] \quad \text{in } \square_{ij}. \tag{3.38}$$

Letting $g = u_x w_x$, we have

$$g_{xx} = u_{xxx} w_x, \qquad g_{yy} = u_{xyy} w_x + 2u_{xy} w_{xy}, \qquad g_{xy} = u_{xxy} w_x + u_{xx} w_{xy}. \tag{3.39}$$

We can apply (3.35) to the integration (2.12), to yield the following bounds through some manipulation:

$$\begin{aligned} \left| \left(\iint_{S_1} - \widehat{\iint_{S_1}} \right) u_x w_x \, \mathrm{d}s \right| &= \left| \sum_{ij} \left(\iint_{\square_{ij}} - \widehat{\iint_{\square_{ij}}} \right) u_x w_x \, \mathrm{d}s \right| \\ &= \sum_{ij} R_{ij}^{(1)} \leqslant C \left\{ h^2 M_3(u) \sum_{ij} h_i k_j \, |w_x(\eta_{ij})| \right. \\ &\quad \left. + \left| \sum_{ij} h_i k_j^3 \, \frac{\partial^2 \tilde{u}_{ij}^{(1)}}{\partial x \partial y} \, w_{xy} \right| + \left| \sum_{ij} h_i^2 k_j^2 \left(\frac{\partial^2 \tilde{u}_{ij}^{(2)}}{\partial x^2} - \frac{\partial^2 \tilde{u}_{ij}^{(3)}}{\partial x^2} \right) w_{xy} \right| \right\}, \end{aligned} \tag{3.40}$$

where $\eta_{ij} \in \square_{ij}$, $\tilde{u}_{ij}^{(k)} = u(\xi_{ij}^{(k)})$, $\xi_{ij}^{(k)} \in \square_{ij}$, $k = 1,2,3$. Bounds of the first term of the rightmost side in (3.40) can be obtained from the Schwarz inequality:

$$\begin{aligned} T_1 &= h^2 M_3(u) \sum_{ij} h_i k_j \, |w_x(\eta_{ij})| \\ &\leqslant h^2 M_3(u) \sum_{ij} h_i k_j \, [|w_x(i+\tfrac{1}{2}, j)| + |w_x(i+\tfrac{1}{2}, j+1)|] \\ &\leqslant C h^2 M_3(u) \overline{|w|}_1 \leqslant C h^2 M_3(u) \, \overline{\|w\|}_1. \end{aligned} \tag{3.41}$$

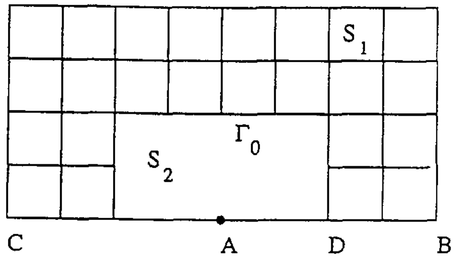

Fig. 2. Partition of Motz's problem with $M_S = 2$.

For the third term in (3.40), we can see from (3.24), (3.38) and the Schwarz inequality,

$$T_{\rm III} = \left|\sum_{ij} h_i^2 k_j^2 \left(\frac{\partial^2 \tilde{u}_{ij}^{(2)}}{\partial x^2} - \frac{\partial^2 \tilde{u}_{ij}^{(3)}}{\partial x^2}\right) w_{xy}\right| \leqslant CM_3(u) h \left|\sum_{ij} h_i^2 k_j^2 w_{xy}\right|$$

$$\leqslant CM_3(u) h \sum_{ij} h_i k_j [|w_{ij} - w_{i+1,j} - w_{i,j+1} + w_{i+1,j+1}|]$$

$$\leqslant CM_3(u) h \sum h_i^2 k_j \left(\frac{|w_{i+1,j} - w_{i,j}|}{h_i} + \frac{|w_{i+1,j+1} - w_{i,j+1}|}{h_i}\right)$$

$$\leqslant CM_3(u) h^2 \; \overline{|w|}_1 \leqslant CM_3(u) h^2 \; \overline{\|w\|}_1. \tag{3.42}$$

Let us now consider the second term of the rightmost side in (3.40):

$$T_{\rm II} = \left|\sum_{ij} h_i k_j^3 \frac{\partial^2 \tilde{u}_{ij}^{(1)}}{\partial x \partial y} w_{xy}\right|$$

$$= \left|\sum_{ij} k_j^2 \frac{\partial^2 \tilde{u}_{ij}^{(1)}}{\partial x \partial y} [w_{ij} - w_{i+1,j} - w_{i,j+1} + w_{i+1,j+1}]\right|$$

$$= \left|\sum_{j} k_j^2 \sum_{i} \frac{\partial^2 \tilde{u}_{ij}^{(1)}}{\partial x \partial y} [w_{ij} - w_{i+1,j} - w_{i,j+1} + w_{i+1,j+1}]\right|. \tag{3.43}$$

Denote $\overline{P_{ij} P_{i,j+1}}$ as a vertical segment of $\partial \square_{ij}$, between the difference vertices (i,j) and $(i,j+1)$. From the assumption that S_1 consists of rectangles only (also see Fig. 2), we may locate the vertical segments $\overline{P_{ij} P_{i,j+1}}$ either inside of S_1 or just on the boundary ∂S_1, i.e.,

Case I: $\overline{P_{ij} P_{i,j+1}} \in S_1$, or

Case II: $\overline{P_{ij} P_{i,j+1}} \in \partial S_1$. Since $\partial S_1 = (\partial S_1 \cap \Gamma) \cup (\partial S_1 \cap \Gamma_0)$, Case II can also be split into following two subcases.

Subcase IIa: $\overline{P_{ij} P_{i\,j+1}} \in \partial S_1 \cap \Gamma$. It is due to the relation $w \in \bar{V}_h$ and the Dirichlet boundary condition (2.2) that

$$\sum_{j} \sum_{\substack{i \\ \text{Case IIa}}} |w_{i,j+1} - w_{ij}| = 0. \tag{3.44}$$

Subcase IIb: $\overline{P_{ij}P_{i\,j+1}} \in \partial S_1 \cap \Gamma_0$. Since

$$|w^-_{i,j+1} - w^-_{ij}| \leqslant |w^+_{i,j+1} - w^+_{ij}| + |w^+_{i,j+1} - w^-_{i,j+1}| + |w^+_{i,j} - w^-_{ij}|, \tag{3.45}$$

we can obtain

$$\begin{aligned}
&\sum_j \sum_{\substack{i\\ \text{Case IIb}}} |w^-_{i,j+1} - w^-_{ij}| \\
&\quad \leqslant \sum_j \sum_{\substack{i\\ \text{Case IIb}}} \{|w^+_{i,j+1} - w^+_{ij}| + |w^+_{i,j+1} - w^-_{i,j+1}| + |w^+_{ij} - w^-_{ij}|\} \\
&\quad \leqslant C\left\{L^\mu \overline{\|w\|}_1 + \frac{1}{h}\overline{\|w^+ - w^-\|_{0,\Gamma}}\right\},
\end{aligned} \tag{3.46}$$

where we have used the following inequality:

$$\begin{aligned}
\sum_j \sum_{\substack{i\\ \text{Case IIb}}} |w^+_{i,j+1} - w^+_{ij}| &= \sum_j \sum_{\substack{i\\ \text{Case IIb}}} \frac{|w^+_{i,j+1} - w^+_{ij}|}{k_j} k_j \\
&\leqslant C\|w^+\|_{1,\Gamma_0} \leqslant CL^\mu \|w\|_{0,\Gamma_0} \leqslant CL^\mu \|w\|_{1,S_2} \leqslant CL^\mu \overline{\|w\|_1}.
\end{aligned} \tag{3.47}$$

Therefore, the second term T_{II} in (3.40) can be reduced from (3.43) to

$$\begin{aligned}
T_{\text{II}} \leqslant \sum_j k_j^2 \Bigg\{ &\sum_{\substack{i\\ \text{Case I}}} |w_{i,j+1} - w_{ij}| \left| \frac{\partial^2 \tilde u^{(1)}_{ij}}{\partial x \partial y} - \frac{\partial^2 \tilde u^{(1)}_{i-1,j}}{\partial x \partial y} \right| \\
&+ \sum_{\substack{i\\ \text{Case IIa}}} \frac{\partial^2 \tilde u^{(1)}_{ij}}{\partial x \partial y} |w_{i,j+1} - w_{ij}| + \sum_{\substack{i\\ \text{Case IIb}}} \frac{\partial^2 \tilde u^{(1)}_{ij}}{\partial x \partial y} |w_{i,j+1} - w_{ij}| \Bigg\}.
\end{aligned} \tag{3.48}$$

For the first term on the right-hand side of (3.48), we obtain also from the Schwarz inequality

$$\begin{aligned}
&\sum_j k_j^2 \sum_{\substack{i\\ \text{Case I}}} |w_{i,j+1} - w_{ij}| \left| \frac{\partial^2 \tilde u^{(1)}_{ij}}{\partial x \partial y} - \frac{\partial^2 \tilde u^{(1)}_{i-1,j}}{\partial x \partial y} \right| \\
&\quad \leqslant CM_3(u)\, h \sum_j k_j^2 \sum_{\substack{i\\ \text{Case I}}} |w_{i,j+1} - w_{ij}| \\
&\quad = CM_3(u)\, h \sum_j k_j^3 \sum_{\substack{i\\ \text{Case I}}} \frac{|w_{i,j+1} - w_{ij}|}{k_j} \\
&\quad \leqslant Ch^2 M_3(u) \overline{|w|}_1 \leqslant Ch^2 M_3(u) \overline{\|w\|}_1.
\end{aligned} \tag{3.49}$$

Combining (3.5), (3.44)–(3.49) yields

$$T_{\text{II}} \leqslant C[h^2 + h^2 L^\mu + h^{1+\sigma/2}] M_3(u) \overline{\|w\|}_h \leqslant C[h^2 L^\mu + h^{1+\sigma/2}] M_3(u) \overline{\|w\|}_h. \tag{3.50}$$

The desired result (3.33) is obtained from (3.40)–(3.42) and (3.50); this completes the proof of (3.30).

For (3.31) we have similarly from Taylor's formula

$$\iint_{\square_{ij}} g\,\mathrm{d}s = \tfrac{1}{4} h_i k_j (g_{ij} + g_{i+1,j} + g_{i,j+1} + g_{i+1,j+1}) + R_{ij}^{(2)}, \tag{3.51}$$

where the truncation errors

$$R_{ij}^{(2)} = -\tfrac{1}{12} h_i k_j \left(h_i^2 \frac{\partial^2 \tilde{g}_{ij}^{(1)}}{\partial x^2} + k_j^2 \frac{\partial^2 \tilde{g}_{ij}^{(2)}}{\partial y^2} \right) - \tfrac{3}{32} h_i^2 k_j^2 \left(\frac{\partial^2 \tilde{g}_{ij}^{(3)}}{\partial x \partial y} - \frac{\partial^2 \tilde{g}_{ij}^{(4)}}{\partial x \partial y} \right). \tag{3.52}$$

Letting $g = fw$, then

$$g_{xx} = f_{xx} w + 2 f_x w_x,\ \ g_{yy} = f_{xy} w + 2 f_y w_y,\ \ g_{xy} = f_{xy} w + f_x w_y + f_y w_x + f w_{xy}. \tag{3.53}$$

Hence, we can obtain similarly

$$\begin{aligned} \left| \left(\iint_{S_1} - \widehat{\iint_{S_1}} \right) f w\,\mathrm{d}s \right| &= \left| \sum_{ij} \left(\iint_{\square_{ij}} - \widehat{\iint_{\square_{ij}}} \right) f w\,\mathrm{d}s \right| \\ &= \sum_{ij} R_{ij}^{(2)} \leqslant C h^2 M_2(f) \sum_{ij} h_i k_j \ [|w(\xi_{ij}^{(1)})| + |w_x(\xi_{ij}^{(2)})| + |w_y(\xi_{ij}^{(3)})|] \\ &\quad + C \left| \sum_{ij} h_i^2 k_j^2\, w_{xy} (f(\xi_{ij}^{(4)}) - f(\xi_{ij}^{(5)})) \right| \leqslant C h^2 M_2(f) \overline{\|w\|}_1. \end{aligned} \tag{3.54}$$

This completes the proof of Lemma 3.3. □

Lemma 3.4. *Let* (3.24) *and* (3.29) *hold. Suppose* (3.14), *i.e.*, S_1 *consists of* $\square_{ij}$, *as well as* $\triangle_{ij}$ *located only close to* ∂S_1. *Then there exist the error bounds*

$$\left| \left(\iint_{S_1} - \widehat{\iint_{S_1}} \right) \nabla u \nabla w\,\mathrm{d}s \right| \leqslant C h^{3/2}\ \overline{\|w\|}_1,\ \ \forall w \in \bar{V}_h, \tag{3.55}$$

$$\left| \left(\iint_{S_1} - \widehat{\iint_{S_1}} \right) f w\,\mathrm{d}s \right| \leqslant C h^{3/2}\ \overline{\|w\|}_1,\ \ \forall w \in \bar{V}_h. \tag{3.56}$$

Proof. It follows that

$$\begin{aligned} \left| \left(\iint_{S_1} - \widehat{\iint_{S_1}} \right) \nabla u \nabla w\,\mathrm{d}s \right| &\leqslant \left| \sum_{\forall ij} \left(\iint_{\square_{ij}} - \widehat{\iint_{\square_{ij}}} \right) \nabla u \nabla w\,\mathrm{d}s \right| \\ &\quad + \left| \sum_{\forall ij} \left(\iint_{\triangle_{ij}} - \widehat{\iint_{\triangle_{ij}}} \right) \nabla u \nabla w\,\mathrm{d}s \right|. \end{aligned} \tag{3.57}$$

We have from (2.16)

$$\left|\sum_{\forall ij}\left(\iint_{\triangle_{ij}} - \widehat{\iint_{\triangle_{ij}}}\right)\nabla u\,\nabla w\,\mathrm{d}s\right| \leqslant CM_2(u)h\sum_{ij}\iint_{\triangle_{ij}}|\nabla w|\,\mathrm{d}s$$

$$\leqslant CM_2(u)h\sqrt{\sum_{ij}|\triangle_{ij}|}\;\overline{|w|}_{1,S_1} \leqslant CM_2(u)h^{3/2}\;\overline{\|w\|}_1. \tag{3.58}$$

In the last inequality of (3.58), we have applied the bounds $\sum_{ij}|\triangle_{ij}| \leqslant Ch$, based on the assumption that all $\triangle_{ij}$ are closed to ∂S_1, where $|\triangle_{ij}|$ denotes the area of triangle $\triangle_{ij}$.

We may follow the proofs in Lemma 3.3, and only notice the different estimates, in particular, those for the first term on the right-hand side of (3.57). In fact, the bounds of (3.43) and (3.48) should be modified as follows:

$$T_{\mathrm{II}} \leqslant CM_3(u)h\sum_j k_j^2 \sum_{\substack{i\\ \text{Case I}^*}} |w_{i,j+1} - w_{i,j}|$$

$$+CM_2(u)\sum_j k_j^2 \sum_{\substack{i\\ \text{Case II}^*}} |w_{i,j+1} - w_{i,j}|$$

$$\leqslant C[h^2M_3(u) + h^{3/2}M_2(u)]\;\overline{\|w\|}_1, \tag{3.59}$$

where Case I* denotes the case where both $P_{i,j}$ and $P_{i,j+1}$ are interior difference nodes, and Case II* the case of either $P_{i,j} \in \partial S_1$ or $P_{i,j+1} \in \partial S_1$.

As a consequence, the bounds (3.55) are proven from (3.57)–(3.59), and (3.32)–(3.43) in Lemma 3.3; the proof of (3.56) is also similar. This completes the proof of Lemma 3.4. □

Note that for $O(h^{3/2})$ in (3.55), we need neither the assumption of the Dirichlet condition (2.2) nor the relation of v^+ and v^- in Case IIb, as used in (3.44) and (3.46) in Lemma 3.3.

Based on Eq. (3.15) and Lemmas 3.1–3.4, we have the following theorem.

Theorem 3.1. *Let all the condition in Lemmas* 3.1–3.4 *hold. Then the solution u_h from the penalty combination* (2.7) *has the error bounds*

$$\overline{\|u_h - u\|}_h \leqslant C\left\{h^t + h^{\sigma/2} + \|R_L\|_{1,S_2\cup S_0^*} + (L^\mu h)^2\right\}, \tag{3.60}$$

where $t = 2$ and $\frac{3}{2}$ for partitions (3.13) *and* (3.14), *respectively.*

Corollary 3.1. *Let all conditions in Theorem* 3.1 *hold. Also suppose that the number L of basis functions used for u^+ is chosen such that $\|R_L\|_{1,S_2\cup S_0^*} = O(h^t)$ and $L = O(|\ln h|)$. Then when $\sigma \geqslant 3$*

$$\overline{\|u_h - u\|}_h = O(h^{3/2}), \tag{3.61}$$

and when $\sigma \geqslant 4$

$$\overline{\|u_h - u\|}_h = O(h^{2-\delta}) \quad \textit{for } (3.13). \tag{3.62}$$

The result (3.62) is a development of superconvergence [5] to quasiuniform rectangles used in S_1 and to the Poisson equation (2.1) as well as the general self-adjoint elliptic equations. The superconvergence rates (3.61) and (3.62) in the norm $\overline{\|\varepsilon\|}_h$ are significant to the penalty combinations. Note that the limitation of $\sigma = 2$ is derived for optimal convergence rates in [6], based on the norm $\|\cdot\|_h$ given in (3.1). This comparison shows that a suitable choice of error norms is important to evaluation of the proposed algorithms. The norm $\overline{\|\varepsilon\|}_h$ is for obtaining superconvergence, but the norm $\|\cdot\|_h$ for obtaining optimal convergence, although the two norms are not equivalent to each other. Overall, the penalty combination is promising in the combined methods.

Remark 1. The penalty coupling techniques can be applied to matching other kinds of numerical methods in [3, 4, 9–16]. For example, if the uniform triangles are located to the boundary ∂S_1, we may use the treatments of [15, p.505] to replace the rather rough approximation (2.16). The solution gradients at the midpoint $(x_{i+1/2}, y_{j+1/2})$ along the slant boundary of $\triangle_{ij}$ are also added into the integration approximation, which can be regarded as the modified finite difference method, or the generalized finite element method. Consequently, the orders in error bounds of Lemma 3.4 and (3.61) in Corollary 3.1 can be improved to regain the superconvergence rate $O(h^{2-\delta})$. In summary, this paper is devoted to study on superconvergence of the combinations using the traditional finite difference methods.

Remark 2. A recent study displays that the penalty combination in this paper can be easily implemented into parallel computing by some embedding techniques, and that parallel algorithms of combinations fall into the framework of the existing domain decomposition methods. Details will appear in [7].

4. Numerical experiments for Motz's problem

In this section, numerical experiments are carried out to verify the superconvergence rate (3.62). Consider the typical Motz problem:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } S, \tag{4.1}$$

$$u|_{x<0 \wedge y=0} = 0, \quad u|_{x=1} = 500, \tag{4.2}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=1} = \left.\frac{\partial u}{\partial y}\right|_{x>0 \wedge y=0} = \left.\frac{\partial u}{\partial x}\right|_{x=-1} = 0, \tag{4.3}$$

where S is a rectangle $(-1 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1)$. The origin $(0,0)$ is a singular point with the solution behaviour $u = O(r^{1/2})$ as $r \to 0$ due to the intersection of the Neumann and Dirichlet

Table 1
Error norms and approximate coefficients by penality combination of RG-FDMs with $P_c = 1$ and $\sigma = 4$

Divisions	$\|\varepsilon^+ - \varepsilon^-\|_{0,\Gamma_0}$	$\|\varepsilon^+ - \varepsilon^-\|_{\infty,\Gamma_0}$	max	$\|\varepsilon\|_{0,S}$	$\|\varepsilon\|_1$	$\overline{\|\varepsilon\|}_1$	$\overline{\|\varepsilon\|}_h$	$\tilde{D}_0$	$\tilde{D}_1$
MS= 2									
$L+1=4$	2.221	3.205	3.449	1.204	21.67	6.294	10.79	399.0352	86.4206
MS= 4									
$L+1=5$	0.4673	0.7261	0.9666	0.2938	10.49	1.658	2.819	400.8677	87.3855
MS= 6									
$L+1=5$	0.1994	0.3144	0.4464	0.1317	6.939	0.7410	1.264	401.0367	87.5252
MS= 8									
$L+1=6$	0.1104	0.1770	0.2787	0.0753	5.191	0.4297	0.7216	401.0926	87.6470
MS= 10									
$L+1=6$	0.0702	0.1125	0.1834	0.0485	4.147	0.2778	0.4645	401.1181	87.6505

conditions. The subdomain S_2 is chosen as a smaller rectangle $(-\frac{1}{2} \leqslant x \leqslant \frac{1}{2},\ 0 \leqslant y \leqslant \frac{1}{2})$, and the admissible functions are

$$v = \begin{cases} v^- = v_1, \\ v^+ = \sum_{\ell=0}^{L} \tilde{D}_\ell\, r^{\ell+1/2} \cos(\ell + \frac{1}{2})\theta, \end{cases} \tag{4.4}$$

where $\tilde{D}_\ell$ are unknown coefficients, and (r, θ) are the polar coordinates with the origin (0,0).

For the sake of simplicity, S_1 in Fig. 2 is split into uniform squares where MS denotes the difference division number along $\overline{\mathrm{DB}}$. Based on the good matching between $L+1$ (the total number of basis functions used) and MS given in [5], we will employ

$$\mathrm{MS} = 2 \text{ and } L+1 = 4; \quad \mathrm{MS} = 4, 6 \text{ and } L+1 = 5; \quad \mathrm{MS} = 8, 10 \text{ and } L+1 = 6. \tag{4.5}$$

Numerical solutions are conducted by penalty combinations of RG-FDMs, and their error norms, as well as the first two important coefficients $\tilde{D}_0$ and $\tilde{D}_1$, are provided in Table 1, where other error norms are defined by:

$$\|\varepsilon\|_{0,S} = \left(\int\!\!\int_S \varepsilon^2 \,\mathrm{d}s \right)^{1/2}, \quad \max = \max_S |\varepsilon|, \tag{4.6}$$

$$\|\varepsilon^+ - \varepsilon^-\|_{0,\Gamma_0} = \left(\int_{\Gamma_0} (\varepsilon^+ - \varepsilon^-)^2 \,\mathrm{d}\ell \right)^{1/2} = \left(\int_{\Gamma_0} (v^+ - v^-)^2 \,\mathrm{d}\ell \right)^{1/2}, \tag{4.7}$$

$$\|\varepsilon^+ - \varepsilon^-\|_{\infty,\Gamma_0} = \max_{\Gamma_0} |\varepsilon^+ - \varepsilon^-| = \max_{\Gamma_0} |v^+ - v^-|. \tag{4.8}$$

The error curves for the results of penalty combination have been depicted in Fig. 3 from Table 1. It is easy to see that when $P_c = 1$ and $\sigma = 4$

$$\overline{\|\varepsilon\|}_1 = \mathrm{O}(h^{2-\delta}), \qquad \overline{\|\varepsilon\|}_h = \mathrm{O}(h^{2-\delta}), \tag{4.9}$$

$$\|\varepsilon\|_1 = \mathrm{O}(h), \qquad \|\varepsilon\|_{0,S} = \mathrm{O}(h^2), \qquad \max = \mathrm{O}(h^{2-\delta}). \tag{4.10}$$

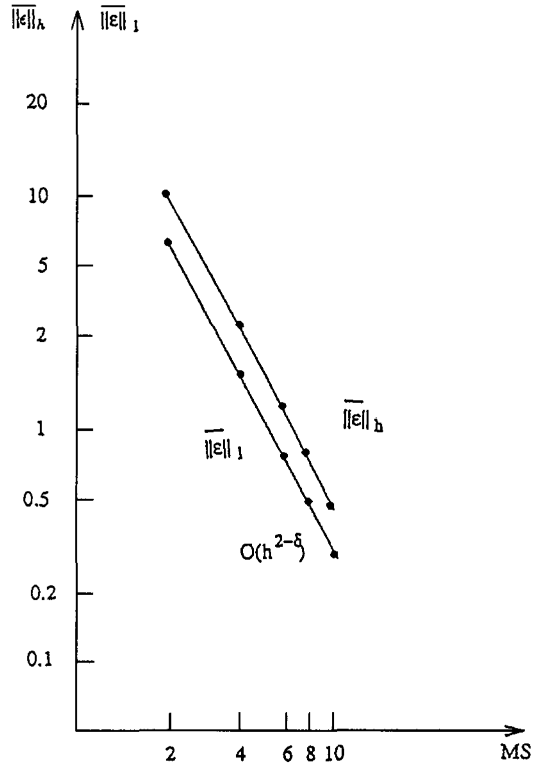

Fig. 3. Curves of the error norms $\overline{\|\varepsilon\|}_h$ and $\overline{\|\varepsilon\|}_1$ from penalty combination of RG-FDMs with $P_c = 1$ and $\sigma = 4$.

Eqs. (4.9) perfectly coincide with the analytical results (3.62). Moreover, the empirical relations (4.10) and the following computational results are also optimal:

$$\|\varepsilon^+ - \varepsilon^-\|_{0,\Gamma_0} = O(h^2), \qquad \|\varepsilon^+ - \varepsilon^-\|_{\infty,\Gamma_0} = O(h^2), \tag{4.11}$$

$$|D_0 - \tilde{D}_0| = O(h^2), \qquad |D_1 - \tilde{D}_1| = O(h^2). \tag{4.12}$$

Using the true coefficients $D_0 = 401.1625$ and $D_1 = 87.6559$, the relative errors of approximate coefficients from the Penalty Combination with $M_S = 10$ and $L + 1 = 6$ are obtained as follows:

$$\frac{|D_0 - \tilde{D}_0|}{|D_0|} = 0.0001, \qquad \frac{|D_1 - \tilde{D}_1|}{|D_1|} = 0.00006. \tag{4.13}$$

being very small indeed. Evidently, penalty combination is simpler and more efficient than nonconforming combination in [5] to solve singularity problems.

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